

Effective de Rham Cohomology - The Hypersurface Case

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ABSTRACT

We prove an effective bound for the degrees of generators of the algebraic de Rham cohomology of smooth affine hypersurfaces. In particular, we show that the de Rham cohomology $H_{\text{dR}}^p(X)$ of a smooth hypersurface X of degree d in \mathbb{C}^n can be generated by differential forms of degree $d^{\mathcal{O}(pn)}$. This result is relevant for the algorithmic computation of the cohomology, but is also motivated by questions in the theory of ordinary differential equations related to the infinitesimal Hilbert 16th problem.

Categories and Subject Descriptors

F.2.2 [Analysis of Algorithms and Problem Complexity]: Nonnumerical Algorithms and Problems—*Geometrical Problems and Computations*

General Terms

Theory

Keywords

de Rham Cohomology, effective bounds, differential forms

1. INTRODUCTION

Let X be a smooth variety in \mathbb{C}^n . A fundamental result of Grothendieck says that the cohomology of X can be described in terms of *algebraic* differential forms on X [13]. More precisely, he proved that the singular cohomology of X is isomorphic to the algebraic de Rham cohomology $H_{\text{dR}}^\bullet(X)$, which is defined as the cohomology of the complex of algebraic differential forms on X . Hence, each cohomology class in $H_{\text{dR}}^p(X)$ can be represented by a p -form

$$\omega = \sum_{i_1 < \dots < i_p} \omega_{i_1 \dots i_p} dX_{i_1} \wedge \dots \wedge dX_{i_p}, \quad (1)$$

where the $\omega_{i_1 \dots i_p}$ are polynomial functions on X . However, Grothendieck's proof gives no information on the degrees of the polynomials $\omega_{i_1 \dots i_p}$. In this paper we prove a bound on

their degrees in the case of a hypersurface. In a forthcoming paper, we will consider the general case.

1.1 Motivation

It is a long standing open question in algorithmic real algebraic geometry to find a single exponential time algorithm for computing the Betti numbers of a semialgebraic set. Single exponential time algorithms are known, e.g., for counting the connected components and computing the Euler characteristic of a semialgebraic set (for an overview see [2], for details and exhaustive bibliography see [3]). The best result in this direction states that for fixed ℓ one can compute the first ℓ Betti numbers of a semialgebraic set in single exponential time [1].

Over the complex numbers, one approach for computing Betti numbers is to compute the algebraic de Rham cohomology. In [21, 25] the de Rham cohomology of the complement of a complex affine variety is computed using Gröbner bases for \mathcal{D} -modules. This algorithm is extended in [26] to compute the cohomology of a projective variety. However, the complexity of these algorithms is not analyzed, and due to their use of Gröbner bases a good worst-case complexity is not to be expected. In [7] a single exponential time (in fact, parallel polynomial time) algorithm is given for counting the connected components, i.e., computing the zeroth de Rham cohomology, of a (possibly singular) complex variety. This algorithm is extended in [24] to one with the same complexity for computing equations for the components. The first single exponential time algorithm for computing *all* Betti numbers of an interesting class of varieties is given in [23]. Namely, this paper shows how to compute the de Rham cohomology of a smooth projective variety in parallel polynomial time. In terms of structural complexity, these results are the best one can hope for, since the problem of computing a fixed Betti number (e.g., deciding connectedness) of a complex affine or projective variety defined over the integers is PSPACE-hard [22].

Besides being relevant for algorithms, our question also has connections to the theory of ordinary differential equations. The long standing infinitesimal Hilbert 16th problem has been solved in [5]. The authors derive a bound on the number of limit cycles generated from nonsingular energy level ovals (isolated periodic trajectories) in a non-conservative perturbation of a Hamiltonian polynomial vector field in the plane. It seems that their proof can be considerably generalized to solutions of certain linear systems of Pfaffian differ-

ential equations. Examples of such systems are provided by period matrices of polynomial maps, once the corresponding Gauss-Manin connexion can be explicitly constructed. For this construction one needs degree bounds for generators of the cohomology of the generic fibers of the polynomial map.

1.2 Known Cases

It follows from the results of [23] that if X has no singularities at infinity, i.e., the projective closure of X in \mathbb{P}^n is smooth, then each class in $H_{\text{dR}}^\bullet(X)$ can be represented by a differential form of degree at most $m(em + 1)d$, where $m = \dim X$, $d = \deg X$, and e is the maximal codimension of the irreducible components of X . However, in general X does have singularities at infinity, and resolution of singularities has a very bad worst-case complexity [4].

Another special case with known degree bounds is the complement of a projective hypersurface, which we will actually use in this paper (see the proof of Theorem 10). The statement follows essentially from [8] and [9], the argument can be found in [10, Corollary 6.1.32]. Let $f \in \mathbb{C}[X_0, \dots, X_n]$ be a homogeneous polynomial, and consider $U := \mathbb{P}^n \setminus \mathcal{Z}(f)$, which is an affine variety. Then, each class in $H_{\text{dR}}^p(U)$ is represented by a (homogeneous) differential form

$$\frac{\alpha}{f^p} \quad \text{with} \quad \deg \alpha = p \deg f$$

(see Section 2 for the definition of the degree of a differential form).

1.3 Main Result

In this paper we prove that each class in the de Rham cohomology $H_{\text{dR}}^p(X)$ of a smooth hypersurface $X \subseteq \mathbb{C}^n$ of degree d can be represented by a differential p -form of degree at most

$$(p+1)(d+1)(2d^n + d)^{p+1} \leq d^{\mathcal{O}(pn)}$$

(see Theorem 10).

2. PRELIMINARIES

An (affine) *variety* in \mathbb{C}^n is the common zero-set

$$X = \mathcal{Z}(f_1, \dots, f_r) = \{x \in \mathbb{C}^n \mid f_1(x) = \dots = f_r(x) = 0\}$$

of a set of polynomials $f_1, \dots, f_r \in \mathbb{C}[X_1, \dots, X_n]$. The *coordinate ring* of a variety X is $\mathbb{C}[X] = \mathbb{C}[X_1, \dots, X_n]/I(X)$, where $I(X) := \{f \in \mathbb{C}[X_1, \dots, X_n] \mid f(x) = 0 \ \forall x \in X\}$ is the *vanishing ideal* of X . Such a \mathbb{C} -algebra is called a (*reduced*) *affine algebra*. By Hilbert's Nullstellensatz, $I(X)$ is the radical of the ideal generated by f_1, \dots, f_r . In particular, if X is a hypersurface defined by one polynomial f , then $I(X) = (g)$, where g is the squarefree part of f , which is also called a *reduced equation* of X . A variety X (and then also its coordinate ring) is called *smooth*, if at any point $x \in X$ we have $\dim_x X = \dim T_x X$, i.e., at x the local dimension and the dimension of the tangent space of X coincide. In particular, a hypersurface is smooth if and only if $\mathcal{Z}(f, \partial_1 f, \dots, \partial_n f) = \emptyset$, where f is a reduced equation of X .

We will also use complete rings, which we get from affine rings by the process of completion [11, Chapter 7]. Let A be an affine algebra and I an ideal in A . The *completion* $\hat{A} = \hat{A}_I$ of A with respect to I is defined as the inverse

limit of the factor rings A/I^ν , $\nu \geq 0$. There is a canonical map $A \rightarrow \hat{A}$, whose kernel is $\bigcap_\nu I^\nu$, so it is injective in our case. Alternatively, if $I = (f_1, \dots, f_r)$, one can define \hat{A} as $A[[T_1, \dots, T_r]]/(T_1 - f_1, \dots, T_r - f_r)$, so its elements are power series in f_1, \dots, f_r [11, Exercise 7.11]. For instance, if $A = B[T]$ and $I = (T)$, then $\hat{A} = B[[T]]$ is the ring of formal power series in T with coefficients in B .

Let A be a \mathbb{C} -algebra (commutative, with 1). The module of *Kähler differentials* $\Omega_A := \Omega_{A/\mathbb{C}}$ is defined as the A -module generated by symbols df for all $f \in A$, modulo the relations of Leibniz' rule and \mathbb{C} -linearity for the *universal derivation* $d: A \rightarrow \Omega_A$. For instance, if $A = \mathbb{C}[X_1, \dots, X_n]$, then Ω_A is the free module with basis dX_1, \dots, dX_n and $df = \sum_i \partial_i f dX_i$.

Now let $\Omega_A^p := \bigwedge^p \Omega_A$ be the p -th exterior power of the A -module Ω_A . We define the differential $d: \Omega_A^p \rightarrow \Omega_A^{p+1}$ by setting $d(fdg_1 \wedge \dots \wedge dg_p) := df \wedge dg_1 \wedge \dots \wedge dg_p$ for $f, g_i \in A$. It is easy to check that d satisfies the graded Leibniz' rule and $d \circ d = 0$. This way we obtain the (*algebraic*) *de Rham complex*

$$\Omega_A^\bullet: A = \Omega_A^0 \xrightarrow{d} \Omega_A^1 \xrightarrow{d} \dots \xrightarrow{d} \Omega_A^n \longrightarrow 0.$$

Its cohomology $H_{\text{dR}}^\bullet(A) := H^\bullet(\Omega_A^\bullet)$ is called the *de Rham cohomology* of A . If $A = \mathbb{C}[X]$ is the coordinate ring of a smooth variety $X \subseteq \mathbb{C}^n$, then $H_{\text{dR}}^\bullet(X) := H_{\text{dR}}^\bullet(A)$ is called the (*algebraic*) *de Rham cohomology* of X . Fundamental for us is the result of [13] stating that the de Rham cohomology $H_{\text{dR}}^\bullet(X)$ of a smooth variety X is naturally isomorphic to the singular cohomology of X .

The module of Kähler differentials of a complete ring may not be finitely generated (see, e.g., [11, Exercise 16.14]). In these cases we use the *universally finite* module of differentials, which is always finitely generated (see [19, §11–12]). Let R be an affine algebra, I an ideal in R , and \hat{R} the completion of R with respect to I . The *completion* of Ω_R with respect to I is $\hat{\Omega}_{\hat{R}} = \hat{R} \otimes_R \Omega_R$ and is called the *universally finite module of differentials* of \hat{R} . There is a *universally finite derivation* $d: \hat{R} \rightarrow \hat{\Omega}_{\hat{R}}$ which is continuous, i.e., it commutes with infinite sums. For instance, for an affine algebra B we have

$$\hat{\Omega}_{B[[T]]} = B[[T]] \otimes \Omega_{B[[T]]} = B[[T]]dT \oplus \Omega_B$$

and $df = \partial_T f dT + d_B f$ for $f \in B[[T]]$, where ∂_T denotes partial derivative with respect to T and d_B denotes the derivation of Ω_B applied to the coefficients of f [19, Example 12.7].

Now let $A = \mathbb{C}[X_1, \dots, X_n]/I$ be an affine algebra. We are interested in degree bounds for the de Rham cohomology of A , so we introduce the following notation. For $f \in \mathbb{C}[X_1, \dots, X_n]$ we denote by \bar{f} its residue class in A . We set

$$\deg \bar{f} := \min\{\deg g \mid g \in \mathbb{C}[X_1, \dots, X_n], \bar{g} = \bar{f}\}.$$

Furthermore, we define the degree of differential forms in Ω_A^p by setting

$$\deg(\bar{f} d\bar{X}_{i_1} \wedge \dots \wedge \bar{X}_{i_p}) := \deg \bar{f} + p.$$

We denote by $\deg(H_{\text{dR}}^p(A))$ the infimum over all integers δ

such that each cohomology class in $H_{\text{dR}}^p(A)$ has a representative of degree $\leq \delta$. For a localization A_f , we define

$$\deg \frac{a}{f^s} := \deg a - s \deg f, \quad \alpha \in \Omega_A^p.$$

3. EFFECTIVE GYSIN SEQUENCE

The main tool in our proof is the Gysin sequence which is the following

LEMMA 1. *Let Y be an irreducible smooth variety and $X \subseteq Y$ a smooth hypersurface. Then there is an exact sequence*

$$\cdots \rightarrow H_{\text{dR}}^p(Y) \rightarrow H_{\text{dR}}^p(Y \setminus X) \xrightarrow{\text{Res}} H_{\text{dR}}^{p-1}(X) \rightarrow H_{\text{dR}}^{p+1}(Y) \rightarrow \cdots$$

A proof of this Lemma is sketched in [20]. Along the same lines we will prove an effective version of it for the case $Y = \mathbb{C}^n$. This means that we will describe the map Res explicitly on the level of differential forms, so that we can control its effect on their degree. Let us first record an easy consequence of the Gysin sequence. Since $H_{\text{dR}}^p(\mathbb{C}^n) = 0$ for $p > 0$, Lemma 1 implies

COROLLARY 2. *For a smooth hypersurface $X \subseteq \mathbb{C}^n$ the residue map*

$$\text{Res}: H_{\text{dR}}^p(\mathbb{C}^n \setminus X) \xrightarrow{\cong} H_{\text{dR}}^{p-1}(X)$$

is an isomorphism for all $p > 0$.

Let $A := \mathbb{C}[X_1, \dots, X_n]$ and $X := \mathcal{Z}(f)$ smooth, where $f \in A$ is squarefree. Then the relevant coordinate rings are

$$\mathbb{C}[\mathbb{C}^n \setminus X] = A_f \quad \text{and} \quad B := \mathbb{C}[X] = A/(f).$$

Furthermore, $d := \deg X = \deg f$. We assume $d \geq 3$.

THEOREM 3. *Let $d \geq 3, p > 0$. The residue map*

$$\text{Res}: H_{\text{dR}}^p(A_f) \rightarrow H_{\text{dR}}^{p-1}(B)$$

is induced by a map $\Omega_{A_f}^p \rightarrow \Omega_B^{p-1}$ which takes a p -form $\omega = \frac{a}{f^s}$ to a $(p-1)$ -form of degree at most

$$(2d^n + d)^s (\deg \omega + sd).$$

EXAMPLE 4. *It is instructive to consider the case $n = 1$. Consider $f = \prod_i^d (X - \zeta_i) \in \mathbb{C}[X]$, and let $\omega = \frac{a}{f^s} dX \in \Omega_{\mathbb{C}[X]_f}$. By expanding the rational function $\frac{a}{f^s}$ into partial fractions and noting that polynomials and the terms of the form $c(X - \zeta_i)^j$ with $j < -1$ can be integrated, one sees that the cohomology $H_{\text{dR}}^1(\mathbb{C} \setminus \mathcal{Z}(f))$ is generated by the differential forms*

$$\frac{dX}{X - \zeta_i}, \quad 1 \leq i \leq d.$$

Since they are also linearly independent, they form a basis of this cohomology. We will see later that the residue map sends $\frac{dX}{X - \zeta_i}$ to $e_i := \prod_{j \neq i} (X - \zeta_j)/a_i$, where $a_i := \prod_{j \neq i} (\zeta_i - \zeta_j)$. Since $\mathcal{Z}(f) = \{\zeta_1, \dots, \zeta_d\}$ and $e_i(\zeta_j) = \delta_{ij}$, the e_i are the idempotents corresponding to the points ζ_i and thus are a basis of the cohomology $H_{\text{dR}}^0(\mathcal{Z}(f))$.

For the proof of Theorem 3 we will need the completion $\hat{A} = \hat{A}_{(f)}$ of the algebra A with respect to the principal ideal (f) . Recall from §2 that $\hat{A} = A[[T]]/(T - f)$, so its elements are power series in f . Note, however, that these power series are not unique. E.g., $f \in A$ can be represented as $f + 0 \cdot f + \cdots$ or $0 + 1 \cdot f + 0 \cdot f^2 + \cdots$. The crucial result for us is a lemma of Grothendieck stating that there is a ring isomorphism $B[[T]] \rightarrow \hat{A}$ (cf. [16, Lemma II,1.2]), which establishes a unique power series representation for the completion. We need to construct this isomorphism explicitly in order to bound degrees. It is easy to come up with a vector space isomorphism, so the difficulty lies in finding a ring isomorphism. The technical construction is in the following statement, which expresses that B is a *formally smooth* \mathbb{C} -algebra [12, Definition 19.3.1].

For a tuple $x = (x_1, \dots, x_n)$ over an affine algebra R we write $\deg(x) := \max_j \deg(x_j)$. If $\psi: R \rightarrow S$ is a homomorphism, we write $\psi(x) := (\psi(x_1), \dots, \psi(x_n))$. For $g \in R$ we denote by \bar{g} its image in any factor algebra of R .

LEMMA 5. *Let $A = \mathbb{C}[X_1, \dots, X_n]$ and $f \in A$ be square-free such that $B = A/(f)$ is smooth. Furthermore, let $\nu \in \mathbb{N}$ and $\psi: B \rightarrow A/(f^\nu)$ be a ring homomorphism that lifts the identity $B \rightarrow B$, i.e., the composition $B \rightarrow A/(f^\nu) \rightarrow B$ is the identity. Then ψ can be lifted to a ring homomorphism $\tilde{\psi}: B \rightarrow A/(f^{\nu+1})$, i.e., the diagram*

$$\begin{array}{ccc} & A/(f^{\nu+1}) & \\ \tilde{\psi} \nearrow & \downarrow \pi & \\ B & \xrightarrow{\psi} & A/(f^\nu) \end{array}$$

commutes. Furthermore, we have

$$\deg(\tilde{\psi}(\bar{X})) \leq d \cdot \deg(\psi(\bar{X})) + d^n.$$

PROOF. Since B is generated as a \mathbb{C} -algebra by the \bar{X}_i , it suffices to define $\tilde{\psi}$ on these elements. Choose $Y_1, \dots, Y_n \in A$ such that $\psi(\bar{X}_i) = \bar{Y}_i$ in $A/(f^\nu)$ for all $1 \leq i \leq n$. Our aim is to define

$$\tilde{\psi}(\bar{X}_i) := \overline{Y_i + a_i f^\nu}, \quad 1 \leq i \leq n, \quad (2)$$

with suitably chosen $a_i \in A$. Then it is clear that $\pi \circ \tilde{\psi} = \psi$. It remains to show that one can define $\tilde{\psi}$ unambiguously by (2). This means that we have to find a_i such that f is mapped to zero in $A/(f^{\nu+1})$. Set $Y := (Y_1, \dots, Y_n)$, $a := (a_1, \dots, a_n)$, and look at the condition

$$f \mapsto \overline{f(Y + af^\nu)} = 0 \quad \text{in} \quad A/(f^{\nu+1}) \quad (3)$$

By the Taylor formula we have

$$f(Y + af^\nu) \equiv f(Y) + \sum_{i=1}^n \partial_i f(Y) a_i f^\nu \pmod{(f^{\nu+1})}.$$

Since $\overline{f(Y)} = \psi(\bar{f}) = 0$ in $A/(f^\nu)$, there exists $p \in A$ such that $f(Y) = pf^\nu$ in A . Furthermore, since $\bar{Y}_i = \bar{X}_i$ in B , condition (3) is satisfied if

$$p + \sum_{i=1}^n \partial_i f a_i \equiv 0 \pmod{(f)}. \quad (4)$$

Since $\mathcal{Z}(f)$ is smooth, we have $\mathcal{Z}(f, \partial_1 f, \dots, \partial_n f) = \emptyset$, so by the effective Nullstellensatz [6, 18, 17] there exist $g_1, \dots, g_n \in A$ such that

$$\sum_{i=1}^n g_i \partial_i f \equiv 1 \pmod{(f)} \quad \text{and} \quad \deg g_i \leq d^n. \quad (5)$$

It follows that (4) can be solved by choosing

$$a_i := -pg_i \quad \text{for all} \quad 1 \leq i \leq n.$$

Furthermore,

$$\deg a_i \leq \deg p + \deg g_i \leq d \deg Y - \nu d + d^n,$$

which implies the claimed degree bound. \square

COROLLARY 6. *There exists an embedding $\psi: B \hookrightarrow \hat{A}$ such that $\psi(\overline{X}_i) = \sum_{\nu=0}^{\infty} a_{i\nu} f^\nu$, where $a_{i\nu} \in A$ with*

$$\deg a_{i\nu} \leq d^n \sum_{i=0}^{\nu-1} d^i + d^\nu \leq d^\nu (2d^{n-1} + 1).$$

PROOF. We start with $\psi_1 := \text{id}_B$ and apply Lemma 5 successively to construct the homomorphisms $\psi_\nu: B \rightarrow A/(f^\nu)$, $\nu \in \mathbb{N}$. Together they define a homomorphism $\psi: B \rightarrow \hat{A}$, which is clearly injective. For $\nu = 0$ we have $\psi_0(\overline{X}) = \overline{X}$, whose degree is 1. By Lemma 5 we have

$$\begin{aligned} \deg a_{i,\nu+1} &\leq d \deg a_{i,\nu} + d^n \leq d(d^n \sum_{i=0}^{\nu-1} d^i + d^\nu) + d^n \\ &= d^n \sum_{i=0}^{\nu} d^i + d^{\nu+1}. \quad \square \end{aligned}$$

For $a \in \hat{A}$ we write $\deg_\nu a \leq \delta_\nu$, if there exists a representation $a = \sum_\nu a_\nu f^\nu$ with $\deg a_\nu \leq \delta_\nu$ for all $\nu \in \mathbb{N}$. The degree bound from the previous corollary reads in this notation $\deg_\nu(\psi(\overline{X}_i)) \leq d^\nu (2d^{n-1} + 1)$.

COROLLARY 7. *For all $a = \sum_\nu a_\nu f^\nu \in \hat{A}$ there exist unique $b \in B$ and $c \in \hat{A}$ such that $a = \psi(b) + cf$ and*

$$\deg b \leq \deg a_0,$$

$$\deg_\nu c \leq \max\{\deg a_{\nu+1}, d^{\nu+1}(2d^{n-1} + 1) \deg a_0\}.$$

PROOF. We have the exact sequence of \hat{A} -modules

$$0 \longrightarrow (f) \longrightarrow \hat{A} \xrightarrow{\pi} B \longrightarrow 0,$$

which splits by the homomorphism ψ . Hence, $\hat{A} \simeq B \oplus (f)$, and the existence and uniqueness of the claimed representation follows. Note that if $a = \psi(b) + cf$, then $b = \pi(a) = \pi(a_0)$. This implies the bound for b . Since ψ and π are homomorphisms and a_0 is a polynomial, we have

$$\psi(b) = \psi(\pi(a_0)) = a_0(\psi(\overline{X})) = a_0(\xi),$$

where we denote $\xi_i := \psi(\overline{X}_i)$ and $\xi = (\xi_1, \dots, \xi_n)$. Hence, $cf = a - a_0(\xi)$ and $\deg_\nu c \leq \max\{\deg_{\nu+1} a, \deg_{\nu+1} a_0(\xi)\}$.

Thus, it remains to bound the degree of $a_0(\xi)$. We can assume that a_0 is a monomial, and for that a straight-forward induction with respect to the degree using Corollary 6 shows

$$\deg_\nu a_0(\xi) \leq d^\nu (2d^{n-1} + 1) \deg a_0,$$

which implies the claimed bound for the degrees. \square

Now we define the homomorphism

$$\hat{\psi}: B[[T]] \rightarrow \hat{A}, \quad \sum_\nu b_\nu T^\nu \mapsto \sum_\nu \psi(b_\nu) f^\nu.$$

LEMMA 8. *The homomorphism $\hat{\psi}$ is an isomorphism, and we have*

$$\deg_\nu \hat{\psi}^{-1}(a) \leq (2d^n + d)^\nu \deg a$$

for all $a \in A$.

PROOF. The injectivity of ψ implies inductively that $\hat{\psi}$ is injective. To show surjectivity, let $a \in \hat{A}$. Construct $\sum_\nu b_\nu T^\nu \in B[[T]]$ with $\sum_\nu \psi(b_\nu) f^\nu = a$. We find the b_ν successively by applying Corollary 7. Let $b_0 \in B$ and $c_0 \in \hat{A}$ with $a = \psi(b_0) + c_0 f$. Then, there is $b_1 \in B$ and $c_1 \in \hat{A}$ with $c_0 = \psi(b_1) + c_1 f$, and so forth. It follows $a = \psi(b_0) + c_0 f = \psi(b_0) + \psi(b_1)f + c_1 f^2 = \dots = \sum_\nu \psi(b_\nu) f^\nu$, which is the image of $\sum_\nu b_\nu T^\nu$. With $\gamma := 2d^{n-1} + 1$ we first prove

$$\deg_\nu c_\mu \leq d^{\nu+\mu+1} \gamma^{\mu+1} \deg a, \quad \nu, \mu \geq 0,$$

by induction on μ . For $\mu = 0$ the claim follows directly from Corollary 7. For $\mu \geq 0$, we have by induction hypothesis

$$\begin{aligned} \deg_\nu c_{\mu+1} &\leq \max\{d^{\nu+\mu+2} \gamma^{\mu+1} \deg a, d^{\nu+1} \gamma d^{\mu+1} \gamma^{\mu+1} \deg a\} \\ &\leq d^{\nu+\mu+2} \gamma^{\mu+2} \deg a, \end{aligned}$$

which proves the claim. Now, again by Corollary 7

$$\deg b_\mu \leq \deg_0 c_{\mu-1} \leq (d\gamma)^\mu \deg a. \quad \square$$

PROOF OF THEOREM 3. Consider the short exact sequence of complexes

$$0 \rightarrow \Omega_A^\bullet \rightarrow \Omega_{A_f}^\bullet \rightarrow \Omega_{A_f/\Omega_A}^\bullet \rightarrow 0.$$

It induces a long exact sequence

$$\begin{aligned} \dots \rightarrow H^p(\Omega_A^\bullet) \rightarrow H^p(\Omega_{A_f}^\bullet) \rightarrow H^p(\Omega_{A_f/\Omega_A}^\bullet) \rightarrow \\ \rightarrow H^{p+1}(\Omega_A^\bullet) \rightarrow \dots \end{aligned}$$

Let $p \geq 1$. We have a map $\lambda: \Omega_B^{p-1} \rightarrow \Omega_{A_f}^p / \Omega_A^p$ which sends the class of ω to $\frac{df}{f} \wedge \omega'$, where $\omega' \in \Omega_A$ is a lift of ω . The residue map $\text{Res}: H_{\text{dR}}^p(A_f) \rightarrow H_{\text{dR}}^{p-1}(B)$ is a cohomology inverse of λ . It can be explicitly described as follows.

By Lemma 8 we have

$$\Omega_{A_f}^\bullet / \Omega_A^\bullet \simeq \hat{\Omega}_{\hat{A}_f}^\bullet / \hat{\Omega}_{\hat{A}}^\bullet \simeq \hat{\Omega}_{B[[T]][T^{-1}]}^\bullet / \hat{\Omega}_{B[[T]]}^\bullet. \quad (6)$$

To construct the image $\text{Res}(\omega)$ of $\omega \in \hat{\Omega}_{B[[T]][T^{-1}]}^p$, expand it in powers of T and extract the coefficient of $\frac{dT}{T}$.

By linearity, it suffices to consider terms of the form

$$\omega = \frac{a}{f^s} dX_{i_1} \wedge \dots \wedge dX_{i_p}, \quad a \in A, \quad i_1 < \dots < i_p, \quad s \geq 1.$$

To construct the image of ω in $\widehat{\Omega}_{B[[T]][T^{-1}]}^\bullet / \widehat{\Omega}_{B[[T]]}^\bullet$ under the isomorphism (6), first note that by Lemma 8 we have

$$b := \widehat{\psi}^{-1}(a) = \sum_{\nu} b_{\nu} T^{\nu} \in B[[T]],$$

$$\Xi_i := \widehat{\psi}^{-1}(X_i) = \sum_{\nu} b_{i\nu} T^{\nu} \in B[[T]],$$

where

$$\deg b_{\nu} \leq (2d^n + d)^{\nu} \deg a, \quad \deg b_{i\nu} \leq (2d^n + d)^{\nu}. \quad (7)$$

Hence, ω is mapped by the isomorphism (6) to

$$\widehat{\omega} = \frac{b}{T^s} d\Xi_{i_1} \wedge \cdots \wedge d\Xi_{i_p}.$$

As noted in §2, we have

$$\begin{aligned} d\Xi_i &= \sum_{\nu \geq 0} (\nu b_{i\nu} T^{\nu-1} dT + d_B b_{i\nu} T^{\nu}) \\ &= \sum_{\nu \geq 0} ((\nu+1)b_{i,\nu+1} dT + d_B b_{i\nu}) T^{\nu}. \end{aligned}$$

The terms of $\widehat{\omega}$ involving dT are of the form

$$\begin{aligned} &\pm (\nu_1 + 1) b_{\mu} b_{i,\nu_1+1} T^{\mu+\nu_1+\cdots+\nu_p-s} \\ &\quad \cdot dT \wedge d_B b_{j_1,\nu_2} \wedge \cdots \wedge d_B b_{j_{p-1},\nu_p} \end{aligned}$$

with some $1 \leq i, j_1, \dots, j_{p-1} \leq n$ and $\mu, \nu_1, \dots, \nu_p \geq 0$. To get the coefficient of dT/T , we have to consider the case $\mu + \nu_1 + \cdots + \nu_p = s - 1$. Using that d_B is of degree 0 together with the estimate (7), it follows that this coefficient is of degree

$$\begin{aligned} &\leq \deg b_{\mu} + \deg b_{i,\nu_1+1} + \deg b_{j_1,\nu_2} + \cdots + \deg b_{j_{p-1},\nu_p} \\ &\leq (2d^n + d)^s (\deg a + p) = (2d^n + d)^s (\deg \omega + sd) \end{aligned}$$

which concludes the proof of the theorem. \square

EXAMPLE 9 (CONT.). We keep the notation of Example 4. To confirm the proposed action of the residue map, consider $\omega = \frac{g}{f^s} dX \in \Omega_{\mathbb{C}[X]_f}$. We claim that

$$\text{Res}(\omega) = \sum_{i=1}^d \text{Res}_{\zeta_i} \left(\frac{g}{f^s} \right) e_i,$$

where Res_{ζ_i} denotes the classical residue at ζ_i of a meromorphic function. Recall that $\text{Res}_{\zeta_i}(h)$ is the coefficient of $(X - \zeta_i)^{-1}$ in the Laurent expansion of h around ζ_i .

According to the proof of Theorem 3, we have to check that

$$\text{Res}(\omega) \cdot \frac{df}{f} \equiv \omega \pmod{d\mathbb{C}[X]_f}.$$

This follows easily from the formulas

- $df = \sum_i a_i e_i dX$,
- $e_i e_j \equiv \delta_{ij} e_i \pmod{(f)}$,
- $\frac{a_i e_i}{f} = \frac{1}{X - \zeta_i}$,
- $\omega \equiv \sum_i \text{Res}_{\zeta_i} \left(\frac{g}{f^s} \right) \frac{1}{X - \zeta_i} dX \pmod{d\mathbb{C}[X]_f}$.

The last identity follows again from the partial fraction decomposition.

Now we are in the position to prove our main result.

THEOREM 10. For each smooth hypersurface $X \subseteq \mathbb{C}^n$ of degree $d \geq 3$ we have

$$\begin{aligned} \deg(H_{\text{dR}}^p(X)) &\leq (p+1)(d+1)(2d^n + d)^{p+1} \\ &\leq d^{\mathcal{O}(pn)}. \end{aligned}$$

PROOF. Let $X = \mathcal{Z}(f)$, where f is squarefree of degree d . If we denote by \tilde{f} the generous homogenization $X_0^{d+1} f(X/X_0)$, then we have $U := \mathbb{C}^n \setminus \mathcal{Z}(f) = \mathbb{P}^n \setminus \mathcal{Z}(\tilde{f})$. As stated in §1.2, each cohomology class in $H_{\text{dR}}^{p+1}(U)$ is represented by a differential form

$$\frac{\tilde{\alpha}}{\tilde{f}^{p+1}}, \quad \deg \tilde{\alpha} = (p+1)(d+1).$$

Dehomogenizing yields a form $\omega = \alpha/f^{p+1}$ with $\deg \alpha \leq (p+1)(d+1)$, hence $\deg \omega \leq p+1$. Since the residue map is surjective, the bound of Theorem 3 implies the claim. \square

EXAMPLE 11 (CONT.). We have seen that in the univariate case $H_{\text{dR}}^0(\mathcal{Z}(f))$ is generated by the idempotents e_i , which are of degree $d-1$. Theorem 10 gives the bound $2d(d+1)$ in this case.

EXAMPLE 12. Consider the hypersurface

$$V = \mathcal{Z}(f) \subseteq \mathbb{C}^2, \quad \text{where } f := XY^2 - X - 1.$$

One easily checks that V is smooth, but has a singularity at infinity, namely $(u : x : y) = (0 : 1 : 0)$. There is one other point at infinity $(u : x : y) = (0 : 0 : 1)$, which is smooth. Topologically, the projective closure \bar{V} is a sphere with two points collapsed (to the singularity), so V is a sphere with three points deleted. It follows that the cohomology is

$$H_{\text{dR}}^0(V) = \mathbb{C} \cdot 1, \quad H_{\text{dR}}^2(V) = 0, \quad \text{and} \quad \dim H_{\text{dR}}^1(V) = 2.$$

Let us find generators of $H_{\text{dR}}^1(V)$. Note that

$$V \rightarrow U := \mathbb{C} \setminus \{\pm 1\}, \quad (x, y) \mapsto y \quad (8)$$

is an isomorphism, and $H_{\text{dR}}^1(U)$ is generated by

$$\frac{dY}{Y-1}, \quad \frac{dY}{Y+1}.$$

The isomorphism (8) identifies X with $\frac{1}{Y^2-1}$ and dX with $-\frac{2Y}{Y^2-1} dY$. Hence, $H_{\text{dR}}^1(V)$ is generated by

$$X(Y+1)dY, \quad X(Y-1)dY.$$

Theorem 10 gives a bound of 3528 for the degrees of generators in this case, so there seems to be room to optimize.

Finally, let us determine the action of the residue map. Its inverse image maps the generators of $H_{\text{dR}}^1(V)$ to

$$\frac{X(Y+1)^2(Y-1)}{f} dX \wedge dY, \quad \frac{X(Y+1)(Y-1)^2}{f} dX \wedge dY,$$

which are cohomologous to

$$\frac{Y+1}{f}dX \wedge dY, \quad \frac{Y-1}{f}dX \wedge dY,$$

and generate $H_{\text{dR}}^2(\mathbb{C}^2 \setminus V)$.

To find the action for general inputs, note that

$$X\partial_X f - f = 1.$$

Hence, any 2-form

$$\frac{h}{f}dX \wedge dY = \frac{h}{f}X\partial_X f dX \wedge dY - h dX \wedge dY$$

is equivalent to

$$\frac{h}{f}X\partial_X f dX \wedge dY = Xh \frac{df}{f} \wedge dY$$

modulo exact forms, and hence is mapped by the residue map to

$$Xh dY.$$

We remark that, also in the general case, one can determine by this method the image under the residue map for forms of order 1 along f without using the completion, but this method does not work for higher orders. This also explains the overestimate of our bound in this example.

Acknowledgements

The author thanks Sergei Yakovenko for asking the question addressed in this paper and bringing to his attention the solution of the infinitesimal Hilbert 16th problem [5]. He is also grateful to the Hausdorff Center for Mathematics, Bonn, for its kind support.

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